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Geometric properties of Dirac fields in a Riemannian space–time: I

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Abstract. Dirac fields in a Riemannian space–time are calculated according to coincidence (type II) or non-coincidence (type I) of two null directions. Type-II fields are then analysed in detail with the help of the Newman–Penrose (NP) formalism. Conditions on the associated energy–momentum tensors are obtained for the various subclasses of the type-II fields.

1. Introduction

The geometric properties of Weyl–Einstein neutrino fields have been analysed in detail by Wainwright (1971). In this and a subsequent work we shall consider the corresponding properties of Dirac and Dirac–Einstein fields in Riemannian space–time manifolds. The discussion falls naturally into two parts, since Dirac fields can be classified, as we shall find, according as two null directions of the field coincide or do not coincide. In the present article we shall confine ourselves to the first case only, which we shall call a Dirac field of type II.

It is necessary to distinguish between test solutions of the field equations and the full Dirac–Einstein system. In the former case it is assumed that the Riemannian space is given and we need to consider only the Dirac equation written in terms of it. In the second case we have to investigate both the gravitational equations of Einstein with the energy–momentum tensor, say, of an electromagnetic field and the Dirac equation. In a curved space–time, the latter can be written locally in the form

$$\gamma^{\alpha}\nabla_{\alpha}\psi+(\mathrm{i}/L)\psi=0 \qquad (\alpha=0,1,2,3) \tag{1}$$

where ∇_{α} is the covariant differential operator, $L=\hbar/m_0c$ the Compton wavelength of a particle of rest mass m_0 and

$$\gamma^{\alpha}=h^{\alpha}_{(j)}\gamma^{j} \qquad (\text{summation over } j=0,1,2,3) \tag{2}$$

where γ^{j} are the usual, flat space–time, Dirac matrices and $h^{\alpha}_{(j)}$ is an orthonormal tetrad. If the Dirac equation (1) is to apply globally (that is, throughout the manifold), global existence of orthonormal tetrads is required. It is known (Geroch 1968, 1970) that the existence of a global, orthonormal tetrad is equivalent to the existence of a spinor structure of the space. Hence we restrict ourselves to space–times which possess this structure. Our results depend on using spinor formalism and the above restriction provides a motivation for our attempt. It is only in this formalism (and, in particular, in its Newman–Penrose version) that the Dirac equation can be analysed conveniently.

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2. Mathematical formalism and classification of Dirac fields

It is well known (van der Waerden 1928, Penrose 1968) that when the γ matrices of Dirac are represented by

$$\gamma^j = \sqrt{2}\begin{pmatrix} 0 & \sigma^j{}_{A\dot{A}} \\ \sigma^{jA\dot{A}} & 0 \end{pmatrix} \qquad j = 0, 1, 2, 3, \tag{3}$$

where $\sigma^j{}_{A\dot{A}}$ are the Pauli matrices (divided by $\sqrt{2}$), the Dirac equation (1) splits into two 2-spinor equations. The σ matrices satisfy

$$\sigma^{kA\dot{A}}\sigma^j{}_{A\dot{A}} = \eta^{kj} = \mathrm{diag}(1, -1, -1, -1) \tag{4}$$

and

$$\sigma^k{}_{A\dot{A}}\sigma_{kB\dot{B}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}. \tag{5}$$

For further details of the notational conventions and spinor calculus, see Pirani (1964) and Penrose (1960). Writing the Dirac bi-spinor ψ as

$$\psi = \begin{pmatrix} u_A \\ \bar{v}^{\dot{B}} \end{pmatrix} \qquad \bar{\psi} = (v^A, \bar{u}_{\dot{B}}) \tag{6}$$

equation (1) becomes

$$\begin{aligned} \nabla^{A\dot{A}}u_A + (\mathrm{i}/\sqrt{2}L)\bar{v}^{\dot{A}} &= 0 \\ \nabla^{A\dot{A}}\bar{v}_{\dot{A}} - (\mathrm{i}/\sqrt{2}L)u^A &= 0 \end{aligned} \tag{7}$$

where

$$\nabla^{A\dot{A}} = \sigma^{\alpha A\dot{A}}\nabla_\alpha \qquad \text{and} \qquad \sigma^{\alpha A\dot{A}} = h^\alpha_{(j)}\sigma^{jA\dot{A}}$$

The Dirac equations (7) can be derived (Hamilton and Das 1977, Hoyle and Narlikar 1974) from the variational principle

$$\delta S = 0 \tag{8}$$

where

$$S = \int \left[\frac{1}{\sqrt{2}L}u_A v^A + \frac{1}{\sqrt{2}L}\bar{u}_{\dot{A}}\bar{v}^{\dot{A}} - \mathrm{i}\bar{u}_{\dot{A}}\nabla^{A\dot{A}}u_A - \mathrm{i}v_A\nabla^{A\dot{A}}\bar{v}_{\dot{A}}\right]\sqrt{-g}\,\mathrm{d}^4x,$$

by varying u and v. Similarly, by varying $\sigma^\alpha{}_{AA}$ or, equivalently, $h^\alpha_{(j)}$, we obtain an energy–momentum tensor

$$\begin{aligned} T_{A\dot{A}B\dot{B}} = \tfrac{1}{4}\mathrm{i}[&u_A\nabla_{B\dot{B}}\bar{u}_{\dot{A}} + u_B\nabla_{A\dot{A}}\bar{u}_{\dot{B}} - \bar{u}_{\dot{A}}\nabla_{B\dot{B}}u_A - \bar{u}_{\dot{B}}\nabla_{A\dot{A}}u_B \\ &+ \bar{v}_{\dot{A}}\nabla_{B\dot{B}}v_A + \bar{v}_{\dot{B}}\nabla_{A\dot{A}}v_B - v_A\nabla_{B\dot{B}}\bar{v}_{\dot{A}} - v_B\nabla_{A\dot{A}}\bar{v}_{\dot{B}}] \end{aligned} \tag{9}$$

which is divergence-free by virtue of equations (7). We can verify also that the vectors

$$j^\alpha = \sigma^\alpha{}_{A\dot{A}}(u^A\bar{u}^{\dot{A}} + v^A\bar{v}^{\dot{A}}) \qquad \text{and} \qquad s^\alpha = \sigma^\alpha{}_{A\dot{A}}u^A\bar{v}^{\dot{A}} \tag{10}$$

are divergence-free. j^α is the Dirac probability current vector, which is always time-like or null. The complex vector s^α is orthogonal to j^α and is null. The real vectors

$$s^\alpha + \bar{s}^\alpha \qquad \text{and} \qquad \mathrm{i}(s^\alpha - \bar{s}^\alpha),$$

which are both divergence-free, are orthogonal to j^α and null only when the latter is null (that is, when u^A is proportional to v^A) but space-like otherwise.

Since the Dirac bi-spinor ψ defines two null directions

$$u_A \bar{u}_{\dot{A}} \qquad \text{and} \qquad v_A \bar{v}_{\dot{A}} \tag{11}$$

respectively, Dirac fields can be classified as follows.

Definition. A Dirac field

$$\psi = \begin{pmatrix} u_A \\ v^{\dot{B}} \end{pmatrix}$$

will be said to be of type I if u_A is not proportional to v_A so that the null directions (11) do not coincide, and of type II if u_A is proportional to v_A and the null directions (11) coincide.

As mentioned in the introduction, only Dirac fields of type II will be considered in this paper. Before discussing them, however, we shall first recast our formalism into Newman–Penrose notation (Newman and Unti 1962, Newman and Penrose 1962, Flaherty 1976) (hereafter denoted as NP).

3. Energy–momentum tensor and Dirac equations in the NP form

The quasi-orthonormal NP tetrad is defined in terms of $h^{\alpha}_{(j)}$ as follows:

$$\begin{aligned} l^{\alpha} &= \frac{1}{\sqrt{2}}(h^{\alpha}_{(0)} + h^{\alpha}_{(3)}) = \sigma^{\alpha A\dot{A}} o_A \bar{o}_{\dot{A}} \\ n^{\alpha} &= \frac{1}{\sqrt{2}}(h^{\alpha}_{(0)} - h^{\alpha}_{(3)}) = \sigma^{\alpha A\dot{A}} i_A \bar{i}_{\dot{A}} \\ m^{\alpha} &= \frac{1}{\sqrt{2}}(h^{\alpha}_{(1)} + \mathrm{i} h^{\alpha}_{(2)}) = \sigma^{\alpha A\dot{A}} o_A \bar{i}_{\dot{A}} \end{aligned} \tag{12}$$

(and $\bar{m}^{\alpha} = \sigma^{\alpha A\dot{A}} i_A \bar{o}_{\dot{A}}$) where the spinor dyad $\{o_A, i_A)$ is normalised so that

$$\epsilon_{AB} = o_A i_B - o_B i_A. \tag{13}$$

In terms of the standard representation of Pauli matrices

$$o^A = \delta^A_0 \qquad i^A = \delta^A_1, \tag{14}$$

the components of the Dirac bi-spinor now become

$$u_0 = u_A o^A \qquad u_1 = u_A i^A \qquad v_0 = v_A o^A \qquad v_1 = v_A i^A. \tag{15}$$

Defining the NP spin coefficients by

$$\begin{aligned} \alpha &= i^A o^{\dot{A}} o^B \nabla_{A\dot{A}} i_B & \beta &= o^A i^{\dot{A}} i^B \nabla_{A\dot{A}} o_B \\ \gamma &= i^A i^{\dot{A}} o^B \nabla_{A\dot{A}} i_B & \epsilon &= o^A o^{\dot{A}} i^B \nabla_{A\dot{A}} o_B \\ \mu &= o^A i^{\dot{A}} i^B \nabla_{A\dot{A}} i_B & \pi &= o^A o^{\dot{A}} i^B \nabla_{A\dot{A}} i_B \\ \rho &= i^A o^{\dot{A}} o^B \nabla_{A\dot{A}} o_B & \tau &= i^A i^{\dot{A}} o^B \nabla_{A\dot{A}} o_B \\ \kappa &= o^A o^{\dot{A}} o^B \nabla_{A\dot{A}} o_B & \nu &= i^A i^{\dot{A}} i^B \nabla_{A\dot{A}} i_B \\ \sigma &= o^A i^{\dot{A}} o^B \nabla_{A\dot{A}} o_B & \lambda &= i^A o^{\dot{A}} i^B \nabla_{A\dot{A}} i_B \end{aligned} \tag{16}$$

and writing

$$D = l^\alpha \nabla_\alpha \qquad \delta = m^\alpha \nabla_\alpha \qquad \Delta = n^\alpha \nabla_\alpha \tag{17}$$

we obtain, after a somewhat lengthy calculation, the Dirac equations (7) in the form

$$\begin{aligned}
&\bar{\delta} u_0 - D u_1 + (\pi - \alpha) u_0 + (\rho - \epsilon) u_1 = -(\mathrm{i}/\sqrt{2}L)\bar{v}_0 \\
&\Delta u_0 - \delta u_1 + (\mu - \gamma) u_0 + (\tau - \beta) u_1 = -(\mathrm{i}/\sqrt{2}L)\bar{v}_1 \\
&\delta \bar{v}_0 - D\bar{v}_1 + (\bar{\pi} - \bar{\alpha})\bar{v}_0 + (\bar{\rho} - \bar{\epsilon})\bar{v}_1 = (\mathrm{i}/\sqrt{2}L) u_0 \\
&\Delta \bar{v}_0 - \delta \bar{v}_1 + (\bar{\mu} - \bar{\gamma})\bar{v}_0 + (\bar{\tau} - \bar{\beta})\bar{v}_1 = (\mathrm{i}/\sqrt{2}L) u_1 .
\end{aligned} \tag{18}$$

Also, the conserved vectors j^α and s^α are

$$j^\alpha = (u_1\bar{u}_1 + v_1\bar{v}_1) l^\alpha + (u_0\bar{u}_0 + v_0\bar{v}_0) n^\alpha - (u_1\bar{u}_0 + v_1\bar{v}_0) m^\alpha - (u_0\bar{u}_1 + v_0\bar{v}_1)\bar{m}^\alpha \tag{19}$$

and

$$s^\alpha = u_0\bar{v}_0 n^\alpha + u_1\bar{v}_1 l^\alpha - u_1\bar{v}_0 m^\alpha - u_0\bar{v}_1\bar{m}^\alpha . \tag{20}$$

From Einstein's equations,

$$G_{\alpha\beta} = -8KT_{\alpha\beta},$$

we can write the energy–momentum tensor (9) as

$$\begin{aligned}
4KT_{\alpha\beta} = {} & \phi_{00} n_\alpha n_\beta - 2\phi_{01} n_{(\alpha}\bar{m}_{\beta)} - 2\phi_{10} n_{(\alpha} m_{\beta)} + \phi_{02}\bar{m}_\alpha \bar{m}_\beta \\
& + \phi_{20} n_\alpha m_\beta + \phi_{11}(4 l_{(\alpha} n_{\beta)} - g_{\alpha\beta}) + \phi_{22} l_\alpha l_\beta + 3\Lambda g_{\alpha\beta}
\end{aligned} \tag{21}$$

where

$$\Lambda = 24R = K(u^A v_A + \bar{u}^A \bar{v}_A)/3\sqrt{2}L \tag{22}$$

and the coefficients ϕ_{AB} are

$$\begin{aligned}
\phi_{00} = {} & 2\mathrm{i}K[u_0 D\bar{u}_0 - \bar{u}_0 D u_0 + \bar{v}_0 D v_0 - v_0 D\bar{v}_0 + (\epsilon - \bar{\epsilon})(u_0\bar{u}_0 - v_0\bar{v}_0) \\
& + \bar{\kappa}(u_0\bar{u}_1 - v_0\bar{v}_1) - \kappa(\bar{u}_0 u_1 - \bar{v}_0 v_1)] \\
\phi_{01} \equiv {} & \bar{\phi}_{\cdot 0} = \mathrm{i}K[u_0 D\bar{u}_1 - \bar{u}_1 D u_0 + u_0\delta\bar{u}_0 - \bar{u}_0\delta u_0 + \bar{v}_1 D v_0 - v_0 D\bar{v}_1 + \bar{v}_0\delta v_0 - v_0\delta\bar{v}_0 \\
& + (\beta - \bar{\alpha} - \bar{\pi})(u_0\bar{u}_0 - v_0\bar{v}_0) + (\bar{\rho} + \epsilon + \bar{\epsilon})(u_0\bar{u}_1 - v_0\bar{v}_1) - \sigma(\bar{u}_0 u_1 - \bar{v}_0 v_1) \\
& - \kappa(u_1\bar{u}_1 - v_1\bar{v}_1)] \\
\phi_{02} \equiv {} & \bar{\phi}_{20} = 2\mathrm{i}K[u_0\delta\bar{u}_1 - \bar{u}_1\delta u_0 + \bar{v}_1\delta v_0 - v_0\delta\bar{v}_1 + (\bar{\alpha} + \beta)(u_0\bar{u}_1 - v_0\bar{v}_1) \\
& - \lambda(u_0\bar{u}_0 - v_0\bar{v}_0) - \sigma(u_1\bar{u}_1 - v_1\bar{v}_1)] \\
\phi_{11} = {} & \tfrac{1}{2}\mathrm{i}K[u_0\Delta\bar{u}_0 - \bar{u}_0\Delta u_0 + u_1 D\bar{u}_1 - \bar{u}_1 D u_1 + u_0\bar{\delta}\bar{u}_1 - \bar{u}_1\bar{\delta} u_0 + u_1\bar{\delta}\bar{u}_0 - \bar{u}_0\delta u_1 \\
& + \bar{v}_0\Delta v_0 - v_0\Delta\bar{v}_0 + \bar{v}_1 D v_1 - v_1 D\bar{v}_1 + \bar{v}_1\bar{\delta} v_0 - v_0\bar{\delta}\bar{v}_1 + \bar{v}_0\delta v_1 - v_1\delta\bar{v}_0 \\
& + (\mu - \bar{\mu} + \gamma - \bar{\gamma})(u_0\bar{u}_0 - v_0\bar{v}_0) + (\bar{\epsilon} - \epsilon + \bar{\rho} - \rho)(u_1\bar{u}_1 - v_1\bar{v}_1) \\
& + (\alpha + \bar{\beta} + \bar{\tau} + \pi)(u_0\bar{u}_1 - v_0\bar{v}_1) - (\bar{\alpha} + \beta + \tau + \bar{\pi})(\bar{u}_0 u_1 - \bar{v}_0 v_1)] \\
\phi_{12} \equiv {} & \bar{\phi}_{21} = \mathrm{i}K[u_0\Delta\bar{u}_1 - \bar{u}_1\Delta u_0 + u_1\delta\bar{u}_1 - \bar{u}_1\delta u_1 + \bar{v}_1\Delta v_0 - v_0\Delta\bar{v}_1 + \bar{v}_1\delta v_1 - v_1\delta\bar{v}_1 \\
& - \bar{\nu}(u_0\bar{u}_0 - v_0\bar{v}_0) - \bar{\lambda}(\bar{u}_0 u_1 - \bar{v}_0 v_1) + (\mu + \gamma + \bar{\gamma})(u_0\bar{u}_1 - v_0\bar{v}_1) \\
& + (\bar{\alpha} - \beta - \tau)(u_1\bar{u}_1 - v_1\bar{v}_1)]
\end{aligned} \tag{23}$$

$$\phi_{22}=2\mathrm{i}K[u_1\Delta\bar{u}_1-\bar{u}_1\Delta u_1+\bar{v}_1\Delta v_1-v_1\Delta\bar{v}_1+\nu(u_0\bar{u}_1-v_0\bar{v}_1)-\bar{\nu}(\bar{u}_0u_1-\bar{v}_0v_1)$$
$$+(\bar{\gamma}-\gamma)(u_1\bar{u}_1-v_1\bar{v}_1)].$$

Because of the Dirac equations we also have

$$\phi_{11}=-3\Lambda+\mathrm{i}K[u_1D\bar{u}_1-\bar{u}_1Du_1+u_0\Delta\bar{u}_0-\bar{u}_0\Delta u_0+\bar{v}_1Dv_1-v_1D\bar{v}_1+\bar{v}_0\Delta v_0-v_0\Delta\bar{v}_0$$
$$+(\gamma-\bar{\gamma})(u_0\bar{u}_0-v_0\bar{v}_0)+(\bar{\tau}+\pi)(u_0\bar{u}_1-v_0\bar{v}_1)-(\tau+\bar{\pi})(\bar{u}_0u_1-\bar{v}_0v_1)$$
$$+(\bar{\epsilon}-\epsilon)(u_1\bar{u}_1-v_1\bar{v}_1)], \tag{24}$$

or a similar expression with the δ operators which we need not write out here.

4. Dirac fields of type II

After the above preliminaries, we are now in a position to discuss the structure of Dirac fields. For a field of type II, that is, when u^A is proportional to v^A, we can choose l^α to be parallel to the null vector $u^A\bar{u}^{\dot{A}}$ (or $v^A\bar{v}^{\dot{A}}$) and o^A to be, equivalently, proportional to u^A. Then, for some complex functions f and g of the coordinates

$$u_A=fo_A \qquad \text{and} \qquad v_A=go_A \tag{25}$$

or

$$u_0=v_0=0 \qquad u_1=f \qquad \text{and} \qquad v_1=g,$$

the Dirac equations (18) now acquire the simplified form

$$\begin{aligned} Df&=(\rho-\epsilon)f \qquad & \delta f&=(\tau-\beta)f+(\mathrm{i}/\sqrt{2}L)\bar{g} \\ Dg&=(\rho-\epsilon)g \qquad & \delta g&=(\tau-\beta)g+(\mathrm{i}/\sqrt{2}L)\bar{f}. \end{aligned} \tag{26}$$

It follows that

$$f=hg$$

where

$$Dh=0 \qquad \text{and} \qquad g\delta h=(\mathrm{i}/\sqrt{2}L)\bar{g}(1-h\bar{h}) \qquad \text{for } f,\, g\neq 0.$$

The vectors j^α and s^α become

$$j^\alpha=(f\bar{f}+g\bar{g})l^\alpha \qquad \text{and} \qquad s^\alpha=f\bar{g}l^\alpha \tag{27}$$

so that j^α is null since l^α is null. It follows that we can expect a Dirac field of type II to exhibit many properties similar to the neutrino fields (Wainwright 1971). The coefficients ϕ_{pq} of (23) now simplify, with the help of the Dirac equation (26), to

$$\begin{aligned} &\phi_{01}=\mathrm{i}K\kappa(g\bar{g}-f\bar{f}) \qquad \phi_{02}=2\mathrm{i}K\sigma(g\bar{g}-f\bar{f}) \\ &\phi_{11}=\mathrm{i}K(\rho-\bar{\rho})(g\bar{g}-f\bar{f}) \\ &\phi_{12}=\mathrm{i}K[f\delta\bar{f}-g\delta\bar{g}+(2\tau-\bar{\alpha})(g\bar{g}-f\bar{f})] \\ &\phi_{22}=2\mathrm{i}K[f\Delta\bar{f}-\bar{f}\Delta f+\bar{g}\Delta g-g\Delta\bar{g}+(\gamma-\bar{\gamma})(g\bar{g}-f\bar{f})] \end{aligned} \tag{28}$$

and, of course,

$$\phi_{00}=\Lambda=0.$$

The energy–momentum tensor takes the form

$$4KT_{\alpha\beta}=-2\phi_{01}n_{(\alpha}\bar{m}_{\beta)}-2\phi_{10}n_{(\alpha}m_{\beta)}+\phi_{02}\bar{m}_\alpha\bar{m}_\beta+\phi_{20}m_\alpha m_\beta$$
$$+\phi_{11}(4l_{(\alpha}n_{\beta)}-g_{\alpha\beta})-2\phi_{12}l_{(\alpha}\bar{m}_{\beta)}-2\phi_{21}l_{(\alpha}m_{\beta)}+\phi_{22}l_\alpha l_\beta. \tag{29}$$

It is now advantageous to introduce subclasses of the type-II fields.

Definition. A Dirac field of type II will be said to be
(i) generic ($\mathrm{II_G}$) if $f\bar{f}\neq g\bar{g}$,
(ii) degenerate ($\mathrm{II_D}$) if $f\bar{f}=g\bar{g}$ (so that $f=\mathrm{e}^{\mathrm{i}\theta}g$ with $D\theta=\delta\theta=0$) and
(iii) totally degenerate ($\mathrm{II_{TD}}$) if $f=g$.

A $\mathrm{II_{TD}}$ field can be regarded as a 'ghost' field since, in this case, the energy–momentum tensor vanishes identically.

We must now digress to note the tetrad transformations allowed by the choice of a fixed l^α. In fact, we have the tetrad freedom of null rotation about l^α, boosts in the l^α–n^α plane and rotations in the m^α–$\bar{m}^\alpha$ plane. These may be written (Flaherty 1976) as

$$\begin{aligned}
l^\alpha&\to\hat{l}^\alpha=l^\alpha\\
m^\alpha&\to\hat{m}^\alpha=m^\alpha+al^\alpha\\
\bar{m}^\alpha&\to\hat{\bar{m}}^\alpha=\bar{m}^\alpha+\bar{a}l^\alpha\\
n^\alpha&\to\hat{n}^\alpha=n^\alpha+\bar{a}m^\alpha+a\bar{m}^\alpha+a\bar{a}l^\alpha
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
l^\alpha&\to\hat{l}^\alpha=A^{-1}l^\alpha\\
m^\alpha&\to\hat{m}^\alpha=\mathrm{e}^{\mathrm{i}\phi}m^\alpha\\
\bar{m}^\alpha&\to\hat{\bar{m}}^\alpha=\mathrm{e}^{\mathrm{i}\phi}\bar{m}^\alpha\\
n^\alpha&\to\hat{n}^\alpha=An^\alpha
\end{aligned}\tag{31}$$

where A and ϕ are real and a is a complex function of coordinates. In terms of the dyad o_A, i_A these transformations are

$$\begin{pmatrix}o^A\\ i^A\end{pmatrix}\to\pm\begin{pmatrix}1&0\\ \bar{a}&1\end{pmatrix}\begin{pmatrix}o^A\\ i^A\end{pmatrix}\tag{32}$$

and

$$\begin{pmatrix}o^A\\ i^A\end{pmatrix}\to\pm\begin{pmatrix}A^{-1/2}\,\mathrm{e}^{\mathrm{i}\phi/2}&0\\ 0&A^{1/2}\,\mathrm{e}^{-\mathrm{i}\phi/2}\end{pmatrix}\begin{pmatrix}o^A\\ i^A\end{pmatrix}\tag{33}$$

with $A>0$. We can note also that under the above tetrad freedom our classification of the type-II fields ($\mathrm{II_G}$, $\mathrm{II_D}$ and $\mathrm{II_{TD}}$) is invariant.

Dirac equations must be compatible with the NP commutators or, in other words, satisfy integrability conditions. Using the equations (Flaherty 1976)

$$\begin{aligned}
\delta\rho&=\bar{\delta}\sigma+\rho(\bar{\alpha}+\beta)-\sigma(3\alpha-\bar{\beta})+(\rho-\bar{\rho})\tau+(\mu-\bar{\mu})\kappa-\psi_1+\phi_{01}\\
D\tau&=\Delta\kappa+(\tau+\bar{\pi})\rho+(\bar{\tau}+\pi)\sigma+(\epsilon-\bar{\epsilon})\tau-(3\gamma+\bar{\gamma})\kappa+\psi_1+\phi_{01}\\
D\beta&=\delta\epsilon+(\alpha+\pi)\sigma+(\bar{\rho}-\bar{\epsilon})\beta-(\mu+\gamma)\kappa-(\bar{\alpha}-\bar{\pi})\epsilon+\psi_1,
\end{aligned}$$

the commutator

$$\delta D - D\delta = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta,$$

we obtain the following conditions on f and g:

$$\begin{aligned} \Delta(\kappa f) - \bar{\delta}(\sigma f) &= f[\kappa(2\gamma + \bar{\gamma} - \bar{\mu}) + \sigma(\bar{\beta} - \bar{\tau} - 2\alpha) - \psi_1] + (\mathrm{i}/\sqrt{2}L)\rho\bar{g} \\ \Delta(\kappa g) - \bar{\delta}(\sigma g) &= g[\kappa(2\gamma + \bar{\gamma} - \bar{\mu}) + \sigma(\bar{\beta} - \bar{\tau} - 2\alpha) - \psi_1] + (\mathrm{i}/\sqrt{2}L)\rho\bar{f}. \end{aligned} \tag{34}$$

It follows that if the null congruence l^α is geodesic and shear-free,

$$\kappa = \sigma = 0,$$

then

$$f\psi_1 = (\mathrm{i}/\sqrt{2}L)\rho\bar{g} \qquad \text{and} \qquad g\psi_1 = (\mathrm{i}/\sqrt{2}L)\rho\bar{f}.$$

Since L is finite and f and g non-zero, we can have either

(i) $\rho = 0$ when also $\psi_1 = 0$

or

(ii) $\rho \neq 0$ when $f\bar{f} = g\bar{g}$ ($\mathrm{II_G}$).

We can state this result as follows.

Theorem 1. For a type-II Dirac field in which the null direction defined by the field is geodesic and shear-free either

(*a*) the null direction is expansion- and twist-free

$$(\psi_1 = 0, \rho = 0)$$

or

(*b*) the null direction has expansion, twist or both and the field is $\mathrm{II_G}$.

In particular, case (*a*) holds for test solutions in a type-D vacuum space–time in which l^α coincides with one of the null directions of the Weyl tensor, and we have the following corollary.

Corollary 1. The only type-D vacuum space–times which admit test Dirac fields of type II in which the null direction of the field is geodesic and shear-free are the 'B' and the 'rotating B' (Kinnersley 1969). We may note that this corollary is considerably more restrictive than the corresponding result for test solutions of the neutrino fields. In the latter case, all type-D vacuum space–times admit solutions in which the null congruence of the field is necessarily geodesic and shear-free (Flaherty 1976, pp 242–50).

5. Conditions on the energy–momentum tensor

We can develop a connection between the various type-II Dirac fields and certain conditions which can be satisfied by the energy–momentum tensor of the field. Because of the similarity of the type-II Dirac field, at least in the $\mathrm{II_G}$ case, to a neutrino field our discussion will closely follow that of Wainwright (1971; see also the review article by Kuchowicz 1974).

The energy density of a field with respect to an observer is defined by

$$E(u) = T_{\alpha\beta}u^{\alpha}u^{\beta}, \tag{35}$$

the velocity vector u^{α} being future-directed ($u_{\alpha}u^{\alpha} = 1$). Similarly, the energy flow vector is given by

$$Q_{\alpha}(u) = T_{\alpha\beta}u^{\beta}. \tag{36}$$

Definition. A field is said to satisfy

(i) the strong energy condition if $E(u) > 0$ and $Q_{\alpha}(u)$ is a future-directed, time-like or null vector, for all observers at each event on an observer's world line for which $T_{\alpha\beta} \neq 0$,

(ii) the weak energy condition of type E_1 if $E(u) \neq 0$, for all observers at each event on an observer's world line for which $T_{\alpha\beta} \neq 0$ and

(iii) the weak energy condition of type E_2 if $Q_{\alpha}(u)$ is a future-directed, time-like or null vector, for all observers at each event on an observer's world line for which $T_{\alpha\beta} \neq 0$.

Let us consider $\mathrm{II_D}$ fields for which

$$f = e^{i\theta}g \qquad \theta \text{ real.}$$

With the help of NP commutators we easily get

$$\begin{aligned} &D(\Delta\theta) = -(\epsilon + \bar{\epsilon})\Delta\theta && \delta(\Delta\theta) = (\tau - \bar{\alpha} - \beta)\Delta\theta \\ &\kappa\Delta\theta = 0 && (\bar{\rho} - \rho)\Delta\theta = 0. \end{aligned} \tag{37}$$

Hence, either

$$\Delta\theta = 0, \tag{38}$$

and (choosing the coordinates and the tetrad in a suitable way) θ is a constant, or $\Delta\theta \neq 0$, and therefore

$$\kappa = 0 \qquad \bar{\rho} - \rho = 0. \tag{39}$$

In this case l^{α} is geodesic and twist-free.

For a $\mathrm{II_D}$ Dirac field we have (according to equation (28))

$$\Lambda = \phi_{00} = 0 \qquad \phi_{01} = \phi_{02} = \phi_{11} = \phi_{12} = 0$$

and

$$\phi_{22} = 4Kg\bar{g}\Delta\theta.$$

Hence the energy–momentum tensor becomes

$$4KT_{\alpha\beta} = \phi_{22}l_{\alpha}l_{\beta} = 4Kg\bar{g}\Delta\theta l_{\alpha}l_{\beta}. \tag{40}$$

Since the velocity of an arbitrary observer has the form

$$u^{\alpha} = pl^{\alpha} + qn^{\alpha} + sm^{\alpha} + \bar{s}\bar{m}^{\alpha},$$

with

$$pq - s\bar{s} = \tfrac{1}{2},$$

it follows that

$$Q_{\alpha}(u) = q\phi_{22}l_{\alpha}.$$

We can summarise these results in the following theorem.

Theorem 2. A Dirac field of type II_D is either a ghost field or it is of energy class E_2. In the latter case,

(a) the energy flow vector is null and parallel to the direction of the field for all observers,

(b) the direction defined by the field is geodesic and twist-free and

(c) there exists a null tetrad in which the energy–momentum tensor assumes the form

$$T_{\alpha\beta}=\phi l_\alpha l_\beta$$

for some function ϕ of the coordinates and, if $\phi>0$, satisfies the strong energy condition.

Thus the non-ghost Dirac fields of type II_D appear as null fields (Wainwright 1971, Penrose 1965, 1969) except that they have necessarily a non-zero rest mass. This, of course, is an important distinction.

Let us now turn our attention to Dirac fields of type II_G for which

$$f\bar{f}\neq g\bar{g}.$$

With the help of Theorem 1, part (a), we can announce the following theorems.

Theorem 3. For a Dirac field of type II_G,

(a) the principal null congruence is geodesic and its shear σ and twist $\omega=\frac{1}{2}\mathrm{i}(\rho-\bar{\rho})$ satisfy

$$\sigma\bar{\sigma}-4\omega^2\leqslant 0$$

if the field is of energy class E_1,

(b) the field is of class E_1 if and only if there exists a null tetrad with respect to which the energy–momentum tensor assumes the form

$$4T_{\alpha\beta}=\phi l_\alpha l_\beta-2(f\bar{f}-g\bar{g})\omega(4l_{(\alpha}n_{\beta)}-g_{\alpha\beta})+2\mathrm{i}(f\bar{f}-g\bar{g})(\bar{\sigma}m_\alpha m_\beta-\sigma\bar{m}_\alpha\bar{m}_\beta)$$

where

$$\phi=2\mathrm{i}[f\Delta\bar{f}-\bar{f}\Delta f+\bar{g}\Delta g-g\Delta\bar{g}+(\gamma-\bar{\gamma})(g\bar{g}-f\bar{f})]$$

and

$$\sigma\bar{\sigma}-4\omega^2\leqslant 0\qquad \phi\omega\leqslant 0,$$

and

(c) the energy density satisfies

$$\operatorname{sgn}E(u)=\operatorname{sgn}[\omega(g\bar{g}-f\bar{f})],$$

providing that

$$\omega\neq 0\qquad\text{and}\qquad |E(u)|\geqslant\tfrac{1}{2}|(g\bar{g}-f\bar{f})\omega|.$$

Theorem 4.

(a) A Dirac field of type II_G is of class E_2 if and only if its principal null congruence is geodesic and shear-free: or

(b) if and only if there exists a null tetrad with respect to which the energy–momentum tensor takes the form

$$T_{\alpha\beta}=\phi l_\alpha l_\beta$$

with ϕ as in theorem 3, part (b).

Theorems 3 and 4 can be proved from (28) and (29) in exactly the same way as in Wainwright's (1971) theorems 3.2, 3.3, 3.4 and 3.5, together with our theorem 1, part (*a*).

We note again that a II_G, E_2 field is also twist- and expansion-free, and satisfies the strong energy condition if and only if $\phi > 0$.

6. Concluding remarks

Our results seem to indicate that Dirac fields of type II represent neutrinos with a non-zero rest mass. Now, the energy–momentum tensor of a null electromagnetic field is of the form

$$T_{\alpha\beta} = \phi l_\alpha l_\beta$$

where $\phi > 0$ and l_α is tangent to the repeated null directions of the field. If the electromagnetic field is source-free, then l_α is necessarily geodesic and shear-free (Robinson–Mariot theorem, see Flaherty (1976) or Robinson (1961).

Hence a gravitational field with a null electromagnetic field of twist-free and geodesic null congruence as source can be regarded also as a gravitational field arising from a non-ghost Dirac field of type II_D satisfying the strong energy condition. This depends, of course, on the Dirac equations being non-trivially satisfied.

Similarly, from theorem 4, part (*b*), it follows that Dirac fields of type II_G and energy class E_2 are null fields. Hence all non-ghost Dirac fields of type II and energy class E_2 are null fields. Furthermore, from theorem 3, part (*a*), and theorem 4, part (*b*), algebraically special gravitational fields, for which the source is a null, source-free electromagnetic field with twist- and expansion-free null congruence, can be regarded as having as source a Dirac field of type II_G which satisfies the strong condition of energy.

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